# Prediction of Permanent Indentation's Effect on Low Velocity Impact Response for Isotropic Plates, Beams, and Rods

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## ABSTRACT

In this study, an essential guidance will be presented for the prediction of maximum impact force, approach velocity and impact duration in every 3 phases of elastic-plastic impact law for isotropic plates, beams, and rods. Dimensionless equations of the motion of the plate were obtained by applying the Reissner-Mindlin plate theory which considers the first-order shear deformation and rotating inertia effects. Then the effects of different dimensionless parameters on force-history have been examined and it is demonstrated that maximum impact force, approach velocity and impact period in every 3 levels of elastic-plastic impact can be predicted by just knowing a non-dimensional parameter. Finally, a criterion necessary for utilizing the effect of permanent indentation on impact response is developed.

**Keywords:** Low velocity impact, Elastic-plastic impact, Permanent indentation

## 1. INTRODUCTION

From past studies, it is well known that plates are susceptible to damage resulting from lateral impact by foreign objects, such as dropped tools, hail and debris thrown up from the runway. Although commercial software is capable of analyzing such impact processes, it often requires extensive skill and rigorous training for design and analysis. Analytical models are useful as they allow parametric studies and provide a foundation for validating the numerical results from large-scale commercial software. Therefore, it is necessary to develop analytical or semi-analytical models to better understand the behaviors of structures under impact and their associated failure process [1]. It should be mentioned that in impact problems where the impact duration is long compared to stress wave periods in the contact region, static contact laws (Hertz law) can be used at moderate impact velocities but if contact stresses are high enough to cause material yielding, permanent indentation as an important parameter must be surveyed [2]. Barnhart and Goldsmith [3] are the first people who investigated the effect of permanent craters on low velocity impact in an experimental study. Then Chattopadhyay [4-6] has surveyed, in a series of studies, the effects of shear deformation and permanent indentation by using plate theory. Impact force and displacement histories were shown to be significantly changed with the inclusion of permanent effects. The Elastic-plastic law that involves permanent indentation in the contact region was presented for isotropic materials by Johnson [2] and for composite materials by Christoforou [7]. Olsson [8] applied the proposed elastic-plastic law for composite material and emphasized that permanent indentation has an incontrovertible effect on the impact response. Chen [9] suggested a new elastic–plastic impact–contact model. The proposed model is applied in the problem of dynamic response of a clamped thin circular plate subjected to a centrally projectile impact.

It is essential to identify the governing non-dimensional parameters that cause the reduction of the independent parameters by combining them in non-dimensional groups reducing the impact parameters and minimizing the number of computations and experiments. This task has been attempted by several researchers [10-12] whose studies are mostly on composite materials. The most important articles in this area were presented by Christoforou and Yigit [13-17]. Yigit and Christoforou [13] developed for the first time a single governing parameter termed the "characteristic impact parameter". They proved that the coefficient of restitution is determined for various impact situations as a function of this single parameter. The authors [14, 15] showed that three non-dimensional parameters are sufficient to govern the low velocity impact response of compact and flexible bodies thoroughly. They also constructed a characterization diagram that shows the functional relationship between the normalized maximum impact force and the two non-dimensional parameters. The characterization diagram provides simple and useful analytical tools for designing experiments and scaling experimental results.

## 2. GOVERNING EQUATIONS OF A MODERATELY THHICK PLATE

Consider a flat, isotropic, moderately thick rectangular plate with thickness $h$, length $a$, and width $b$, modulus of Elasticity $E_p$, Poisson's ratio $\nu_p$, and mass $m_p$, oriented so that its mid-plane surface contains the $x_1$ and $x_2$-axis of a Cartesian coordinate system $(x_1, x_2, x_3)$ as illustrated in Figure 1.Two edges of the plate parallel to the $x_2$-axis are assumed to have simply supported boundary condition while the other edges can be a combination of free, simply supported, and clamped boundary conditions. In the absence of $p$ that is any applied transverse point or distributed loads opposing the $x_3$-direction and assuming free harmonic motion, the governing differential equations of motion were derived [18] based on Mindlin first order shear deformation theory

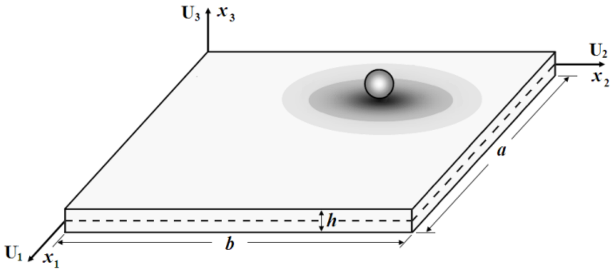



Figure 1.Moderately thick rectangular plate under transverse impact

$$D\left[\nu_1(\psi_{1,11}+\psi_{1,22})+\nu_2(\psi_{1,11}+\psi_{2,12})\right]-\kappa^2Gh(\psi_1-\psi_{3,1})=-(1/12)\rho h^3\ddot{\psi}_1 \quad (1-a)$$

$$D\left[\nu_1(\psi_{2,11}+\psi_{2,22})+\nu_2(\psi_{1,12}+\psi_{2,22})\right]-\kappa^2Gh(\psi_2-\psi_{3,2})=-(1/12)\rho h^3\ddot{\psi}_2 \quad (1-b)$$

$$\kappa^2Gh\left[\psi_{3,11}+\psi_{3,22}-\psi_{1,1}-\psi_{2,2}\right]=\rho h^3\ddot{\psi}_3 \quad (1-c)$$

In which $\rho$ is mass density per unit volume, $D=(E_p h^3)/\left(12\left(1-\nu_p{}^2\right)\right)$ is flexural rigidity, $G=1/\left(2\left(1+\upsilon_p\right)\right)$ is shear correction factor that is applied to transverse shear forces since they have a nearly parabolic dependency on the thickness coordinate, and In Eq. (1) and in the following part of the present paper, the symbol '',’’ is used to indicate derivative; e.g. $\psi_{3,11}$ is equivalent to $\partial^2\psi_3/\partial^2 x_1$ while $\psi_{3,2}$ is simply $\partial\psi_3/\partial x_2$. The $\upsilon_1$, $\upsilon_2$ are defined as

$$\upsilon_1=\left(1-\upsilon_p\right)/2,\ \upsilon_2=\left(1+\upsilon_p\right)/2 \quad (2)$$

For obtaining dimensionless equations these dimensionless parameters are utilized

$$X_1=x_1/a,\ X_2=x_2/a,\ \overline{\psi}_1=a\psi_1/\alpha_1,\ \overline{\psi}_2=a\psi_2/\alpha_1\ \overline{\psi}_3=\psi_3/\alpha_1,\ \delta=h/a,$$
$$\eta=a/b,\ \beta=\omega/\varphi,\ \tau=\varphi t \quad (3)$$

Where $\beta$ is frequency parameter. Also for obtaining the frequency of free vibration of plate under parameters should be defined

$$\overline{\psi}_1(X_1,X_2,\tau)=\tilde{\psi}_1(X_1,X_2)e^{i\beta\tau} \quad (4-a)$$
$$\overline{\psi}_1(X_1,X_2,\tau)=\tilde{\psi}_1(X_1,X_2)e^{i\beta\tau} \quad (4-b)$$
$$\overline{\psi}_1(X_1,X_2,\tau)=\tilde{\psi}_1(X_1,X_2)e^{i\beta\tau} \quad (4-c)$$

By using dimensionless parameters that are defined in equations 3 and 4-a to 4-c, equations 1-a to 1-c can be rewritten as

$$\lambda\left[\tilde{\psi}_{1,11}+\eta^2\tilde{\psi}_{1,22}+(\nu_2/\nu_1)\left(\tilde{\psi}_{1,11}+\eta\tilde{\psi}_{2,12}\right)\right]-(12\kappa^2\lambda/\delta^2)\left(\tilde{\psi}_1-\tilde{\psi}_{3,1}\right)=-\beta^2\tilde{\psi}_1 \quad (5-a)$$
$$\lambda\left[\tilde{\psi}_{2,11}+\eta^2\tilde{\psi}_{2,22}+(\nu_2/\nu_1)\eta\left(\tilde{\psi}_{1,12}+\eta\tilde{\psi}_{2,22}\right)\right]-(12\kappa^2\lambda/\delta^2)\left(\tilde{\psi}_2-\tilde{\psi}_{3,2}\right)=-\beta^2\tilde{\psi}_2 \quad (5-b)$$
$$\left(\kappa^2\lambda\right)\tilde{\psi}_{3,11}+\left(\kappa^2\lambda\right)\eta^2\tilde{\psi}_{3,22}-\kappa^2\lambda\left(\tilde{\psi}_{1,1}+\eta\tilde{\psi}_{2,2}\right)=-\beta^2\tilde{\psi}_3 \quad (5-c)$$

Where $\lambda = (12D\nu_1)/(\rho h^3 a^2 \varphi^2)$ is the impact parameter. To obtain solution of these equations, the exact vibrational solution were used which has been presented in refrence [18] (this solution has been given for six so-called boundary condition including SSSS, SSSC, SCSC, SSSF, SFSF, and SCSF).

For having the forced response of the plate compared to the imposed force, Eigenvalue Expansion method is used. Finally, after using mentioned method, the plate's movement in the direction of thickness will be as follows

$$\overline{\psi}_3(X_1, X_2, \tau) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\overline{\overline{\psi}}_3^{mn}(X_1, X_2)}{K^{mn}\beta^{mn}} \int_0^{\tau} Q^{mn}(\Gamma) Sin\left[\beta^{mn}(t-\tau)\right] d\Gamma \tag{6}$$

By considering

$$K^{mn} = \int_0^1\int_0^1 \left[(\delta^2/12)\left(\left(\overline{\overline{\psi}}_1^{mn}\right)^2 + \left(\overline{\overline{\psi}}_2^{mn}\right)^2\right) + \left(\overline{\overline{\psi}}_3^{mn}\right)^2\right] dX_1 dX_2 \tag{7}$$

$$Q^{mn}(\tau) = \int_0^1\int_0^1 \overline{p}\,\overline{\overline{\psi}}_3 \, dX_1 dX_2 \tag{8}$$

In the above equation, $\overline{p}(X_1, X_2, \tau) = \dfrac{p(x_1, x_2, t)}{\rho h \varphi^2 \alpha_{\max}}$ is the non-dimensional transverse force. Also, $\overline{\overline{\psi}}_1^{mn}(X_1, X_2), \overline{\overline{\psi}}_2^{mn}(X_1, X_2)$, and $\overline{\overline{\psi}}_3^{mn}(X_1, X_2)$ are the vibrational modeshape functions, n, m are the numbers of semi-waves in the directions $X_1$, $X_2$.

## 3. NORMALIZATION OF IMPACT LAWS

### Hertz Theory

In impact problems where the impact duration is long compared to wave periods, static contact laws such as those given in the following can be used in impact problems at moderate impact velocities [2]. In Hertz theory, the relationship between force and indentation in the contact of two elastic bodies are as follows

$$F = k_2 \alpha^{(3/2)} \quad 0 \le \alpha \le \alpha_{\max} \tag{9}$$

In which

$$k_2 = (4/3)\sqrt{R_i}E^*, \quad 1/E^* = \left(1-\upsilon_i^2\right)/E_i + \left(1-\upsilon_p^2\right)/E_p \tag{10}$$

In the above-mentioned formula, $\upsilon_p$, $E_p$ are Poisson's ratio and Young's modulus of plate respectively and $\upsilon_i$ and $E_i$ are Poisson's ratio and Young's modulus of impactor respectively and $\alpha$ is the indentation. Now suppose that a spherical object with initial velocity of $V_0$ impacts a static plate, in which based on Newton's law and by knowing $\ddot{\alpha} d\alpha = \dot{\alpha} d\dot{\alpha}$ and when the indentation is maximum, relative velocity of $\dot{\alpha}$ is zero; we have

$$\alpha_{\max} = \left( \frac{5 m_i}{4 k_2} V_0^2 \right)^{2/5} \tag{11}$$

That is the precise upper limit of low velocity impact [19]. Now by supposing $\psi_i$ movement for the impact object and equation 3, we will have

$$\overline{\psi}_i = \psi_i / \alpha_{\max}, \tau = \varphi t \tag{12}$$

By supposing equations 12, we have

$$\overline{\psi}_i = -F(t)/(m_i \alpha_{\max} \varphi^2) = -\overline{F}(\tau) \tag{13}$$

Now if we suppose that $\overline{\alpha} = \alpha / \alpha_{\max}$, relation 13 will be given as follows

$$\overline{F}(\tau) = k_2 \alpha^{(3/2)} /(m_i \alpha_{\max} \varphi^2) = k_2 \sqrt{\alpha_{\max}} \overline{\alpha}^{(3/2)} /(m_i \varphi^2) \tag{14}$$

That would be given from relation 14

$$\varphi^2 = k_2 \sqrt{\alpha_{\max}} / m_i \tag{15}$$

By supposing relation 15, the force relation in the elastic phase would be given as follows

$$\overline{F}(\tau) = (\overline{\alpha})^{(3/2)} \quad 0 \le \overline{\alpha} \le 1 \tag{16}$$

**Elastic-Plastic Impact Law**

If the stress made of the impact between the object and the plate is more than the Yield strength of the plate, the Hertz impact theory can no longer give a proper response and the permanent indentation should also be taken into account.

Based on elastic-plastic contact law [2], it is supposed that impact duration includes 3 phases

1- Initial elastic indentation that follows Hertz law, so in this phase, the relation of $F = k_2 \alpha^{(3/2)}$ is right. This phase continues until the stress gets to $S_y$ (Yield strength).
2- The indentation that is started from the end of first phase and during which a surrounded plastic zone by an elastic ring grows from the center of the contact area toward its outer part under Yield strength.
3- The Restitution that starts by making the relative velocity of the two objects' impact zero and during which, the plastic zone that is made at the end of the second phase which is surrounded by elastic ring, returns plastically under the ending elastic Yield strength in the second phase. The radius of permanent indentation of the ring at the end of this phase is equal to the radius of the plastic circle which is made at the end of the second phase.

Based on the given relations in [17], let us turn to normalization of the three phases. The elastic phase follows Hertz law, so that the relationship between the force and indentation is the same as relation 9 and only differs in

$$F = k_2 \alpha^{(3/2)} \quad 0 \le \alpha \le \alpha_1, \quad \alpha_1 = \frac{\pi^2 R_i S_y^2}{4(E^*)^2} \tag{17}$$

And the non-dimensional equation would be the same as relation 16

$$\bar{F}(\tau) = (\bar{\alpha})^{(3/2)} \quad 0 \le \bar{\alpha} \le 1 \tag{18}$$

In the elastic-plastic phase, based on relations in [17], the given dimensional force would be as follows

$$\bar{F}(\tau) = (3\bar{\alpha} - 1)/2 \qquad 1 \le \bar{\alpha} \le \bar{\alpha}_2 \tag{19}$$

In the loading elastic phase, we also have a relation in non-dimensional form as follows

$$\bar{F}(\tau) = \frac{3}{2}\left[(\bar{\alpha}_2 - 1)\sqrt{\bar{\alpha} - \bar{\alpha}_2 + 1} + \frac{2}{3}\sqrt[3/2]{\bar{\alpha} - \bar{\alpha}_2 + 1}\right] \tag{20}$$

**4. FORCE HISTORY DETERMINATION WITH SMALL TIME INCREMENT METHOD**

After having the plate's response compared to the force of $\bar{p}(X_1, X_2, \tau)$ as expansion of modes and force in the every 3 phases, we will get to the transverse impact of the impactor on the rectangular isotropic plate. By considering $X_1 = \varphi, X_2 = \zeta$ and $\bar{p}(\phi, \zeta, \tau)$ as the impacting point of the impactor to the plate, $\bar{p}(\phi, \zeta, \tau)$ will be the imposed force on the plate by impactor that as it is considered in the following relationship, $\bar{p}(\phi, \zeta, \tau)$ force is known as the impact function

$$\bar{p}(\varphi, \zeta, \tau) = \varphi\varphi\, \bar{F}(\tau)\, \delta(X_1 - \varphi)\, \delta(X_2 - \zeta), \varphi\varphi = m_i / m_p \tag{21}$$

$$\begin{aligned} \bar{\psi}_{3p}(X_1, X_2, \tau) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\varphi\varphi\, \bar{\bar{\psi}}_{3p}^{mn}(X_1, X_2)}{K^{mn}\beta^{mn}} \int_0^{\tau} \bar{F}(\tau) Sin\left[\beta^{mn}(\tau - \Gamma)\right] d\Gamma \\ * \int_0^1\int_0^1 \delta(X_1 - \varphi)\, \delta(X_2 - \zeta) \bar{\bar{\psi}}_{3p}^{mn}(X_1, X_2) dX_1 dX_2 \end{aligned} \tag{22}$$

Based on the given feature for the impact function, we have

$$\int_0^{\varphi}\int_0^{\zeta} \delta(X_1 - \phi)\delta(X_2 - \zeta)\bar{\bar{\psi}}_{3p}^{mn}(X_1, X_2) dX_1 dX_2 = \bar{\bar{\psi}}_{3p}^{mn}(\phi, \zeta) \tag{23}$$

And finally, the plate's movement relationship toward $X_3$ due to stimulation of $\bar{p}(X_1, X_2, \tau)$ force at the point of $X_1 = \varphi, X_2 = \zeta$ in every point would be as follow

$$\bar{\psi}_{3p}(\phi, \zeta, \tau) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\varphi\varphi\, \bar{\bar{\psi}}_{3p}^{mn}(X_1, X_2)\, \bar{\bar{\psi}}_{3p}^{mn}(\phi, \zeta)}{K^{mn}\beta^{mn}} \int_0^{\tau} \bar{F}(\tau) Sin\left[\beta^{mn}(\tau - \Gamma)\right] d\Gamma \tag{24}$$

The given $\lambda$ considers separately many of the given properties such as $E_p / S_y$ and $h / R_i$. Now by considering the Eq. (12), we have

$$\overline{\psi}_i(0) = 0, \quad \varsigma = \left.\frac{d\overline{\psi}_i(t)}{dt}\right|_{t=0} = \frac{V_0}{\varphi\,\alpha_1} \tag{25}$$

That is elastic impact, $\xi$ is equal to $2/\sqrt{5}$ that does not depend on the initial velocity of the impactor. By using the given result from the Newton laws for the impactor and its Laplace solution, we will have

$$\psi_i(\tau) = \zeta\tau - \int_0^{\tau} (\tau - \Gamma) F \Gamma \tag{26}$$

By considering that indentation is given in every moment of the relation $\overline{\alpha} = \overline{\psi}_i - \overline{\psi}_{3p}$, we will have

$$\overline{\alpha} = \zeta\tau - \int_0^{\tau} \overline{F}(\Gamma)(\tau - \Gamma)d\Gamma - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \varphi\varphi(\overline{\overline{\psi}}_{3p}(\phi,\zeta))^2 / (K^{mn}\beta^{mn}) \int_0^{\tau} \overline{F}(\Gamma) Sin\left[\beta^{mn}(\tau - \Gamma)\right] d\Gamma \tag{27}$$

By replacing relation 27 in relations 18, 19 and 20, the impact force can be calculated in every 3 phases of Elastic-plastic impact law. Finally, for separating obtaining relations small-time increment method is used in which the contact force within the small-time increment of $\Delta\tau$ is assumed constant and $\Delta\tau$ is a small fraction of the vibrational period time of the plate.

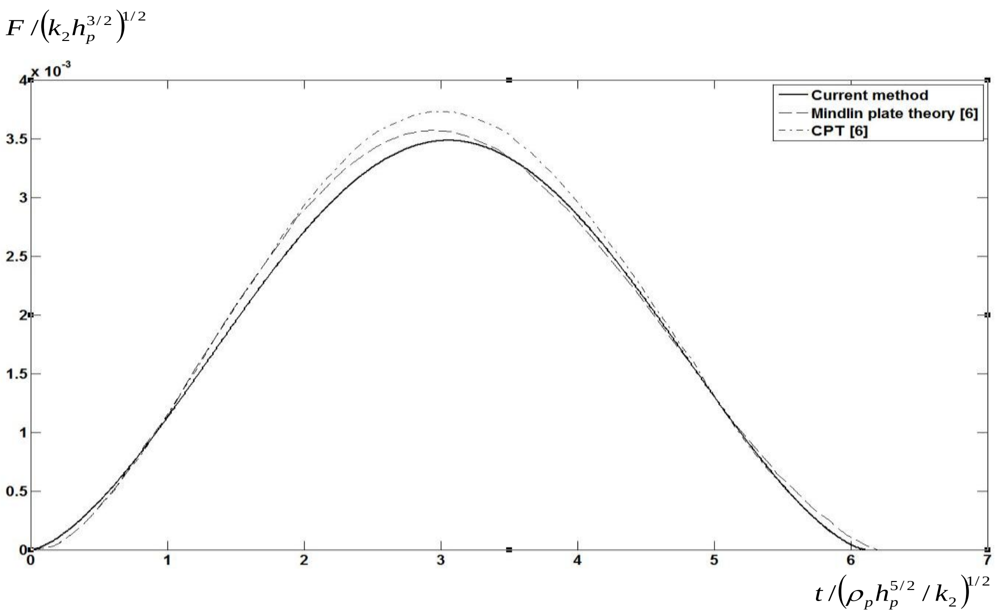
## 5. NUMERICAL EXAMPLES AND DISCUSSION

The main reference for surveying the given method in the study is the mentioned article by chattopadhyoy and Saxena [6]. In the article, the effect of two important mechanisms, that is, shear deformation and permanent indentation on the response of impact has been surveyed in elastic plate. In the given article, for solving displacement equations resulted from classic plate theory and also Mindlin theory (without attention to rotational inertia effect) power series method has been used and finally, the effects of two given mechanisms have been presented in force history figure. The plate and impactor properties have been presented in accordance with the given article in Table 1.

Table 1.The plate and impactor properties

| plate | impactor |
|---|---|
| $E_p = 206.85\,GPa$<br>$\nu_p = 0.3$<br>$\rho_p = 7837\ kg/m^3$<br>$a,b,c: 0.762, 0.0127\ m$ | $E_i = 206.85\,GPa$<br>$\nu_i = 0.3$<br>$\rho_i = 7837\ kg/m^3$<br>$R_i: 0.0127\ m$<br>$V_0 = 45.72\ MPa$ |

(a)




(b)

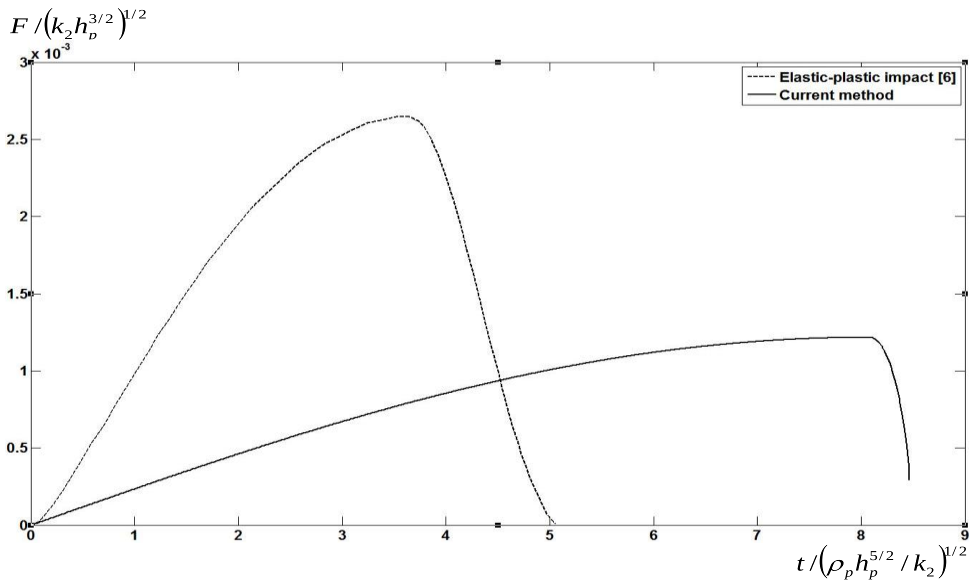



Figure 2.(a) Comparison results obtained from exact solution of Mindlin theory and results from MPT and CPT obtained in [6], (b) Comparing impact force history with the one in [6]

As can be seen in Figure 2-a, using exact solution method with the consideration of the effect of shear deformation and also the effect of rotational inertia will lead to a decrease of %2.749 in maximum impact force resulted from Hertz law in comparison with the force history resulted from Mindlin's plate theory in reference [6]. In Figure 6-b, in addition to the effect of shear deformation, the effect of permanent indentation will also be examined on the given force history and has been compared with the identical example in the mentioned article. The reason for the intense decrease of maximum impact force and the increase in the impact period is related to using different elastic-plastic impact laws.

Most of the effective parameters on the impact force history by considering permanent indentation like $\delta$ ،$\eta$ and the ratio of the impactor's mass to the plate's mass, have little effect on force history, so we ignored considering the evaluation of their effects in this section and only the effects of $\lambda$ and $\xi$ are given in Figures 3 and 4. Notice that the following Figures have properties mentioned in Table 2. Figure 3 shows that by increasing the impact parameter, the maximum force, also, increases, but the impact duration decreases. In addition, as it is clear, the impact parameter related to the elastic-plastic impact, puts most of the effective non-dimensional parameters in their places like $\frac{E_p}{S_y}$.

Table 2.Assumed Material properties for convergency

| $\eta$ | $\delta$ | $\upsilon_p$ | $\lambda$ | $\kappa^2$ | $\xi$ | $\varphi\varphi$ |
|---|---|---|---|---|---|---|
| 0.4 | 0.1 | 0.3 | 0.5 | 0.86667 | 100 | 0.01 |

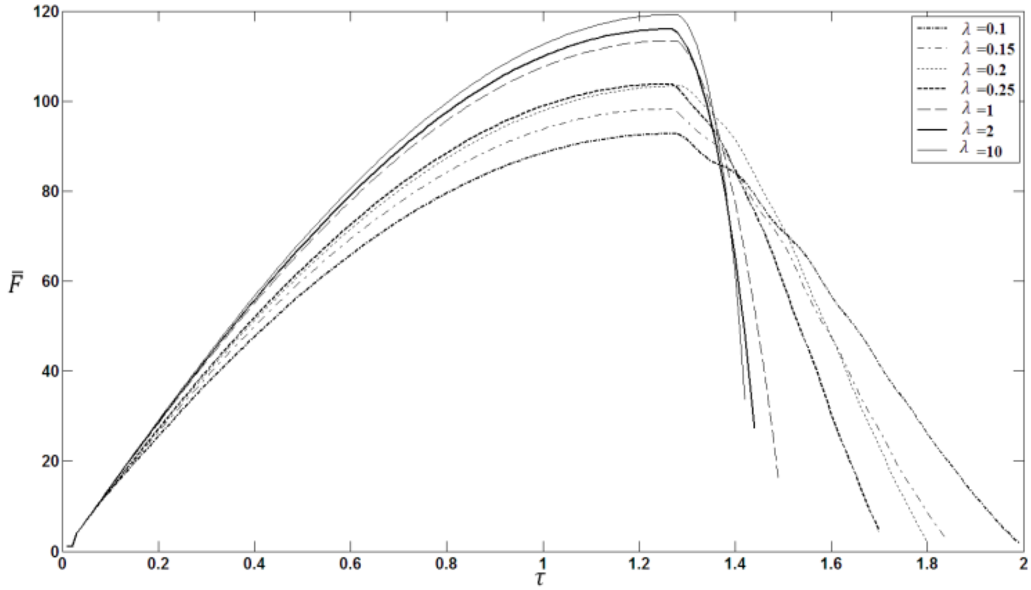



Figure 3. The effect of impact parameter on elastic-plastic impact force history

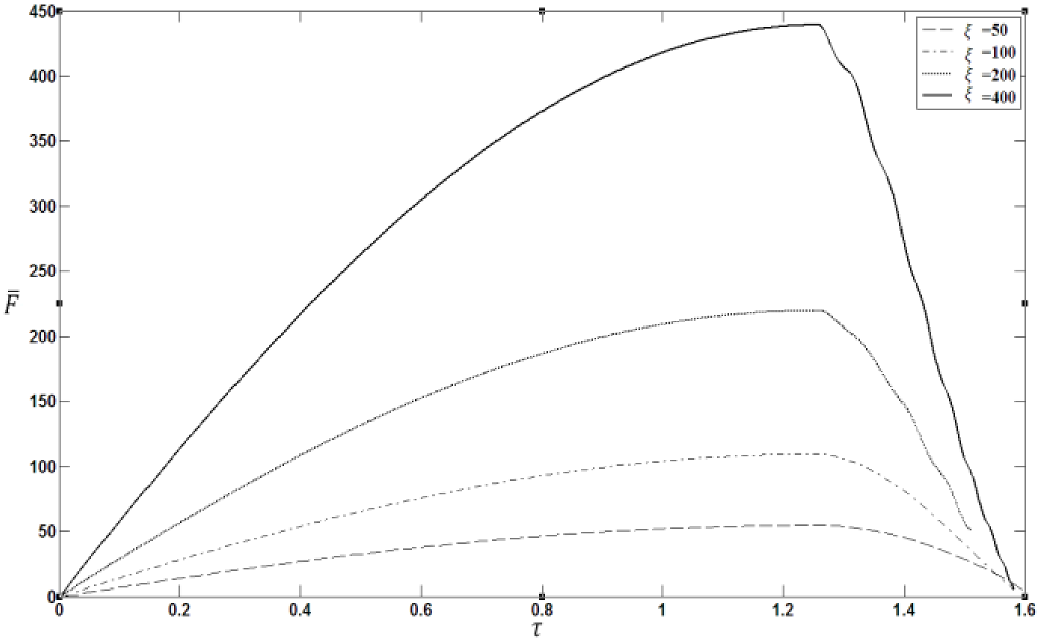



Figure 4. The effect of non-dimensional velocity on elastic-plastic impact force history

Covering all possible impact attitudes in one experimental program is very difficult. Hence, the reduction of the independent impact parameters to only one non-dimensional parameter, not only gives an appropriate physical vision but it causes a valuable tool for generalizing impact results through the use of the minimum amount of data and model test. In fact, only by knowing the plate and impactor properties, can we predict impact force, impact period and approach velocity values without running a highly complicated simulation or doing a test. Among the parameters, $\xi$ has the greatest effect on the impact response. In Figure 4, the effect of the changes of this parameter and its effects on the impact history are studied. In fact, increase in this parameter causes great increase in the impact force and decrease in impact duration, compared to the other parameters. Now in this part, after investigating all non-dimensional parameters on force history proved that maximum impact force, approach velocity and impact period in every 3 levels of elastic-plastic impact can be predicted by knowing only a non-dimensional parameter, $\xi$. By defining $\bar{\psi}_i = \dfrac{\psi_i}{\alpha_{\max}}$, $\tau = \varphi t$, $\bar{\psi}_i = \dfrac{V_0}{\varphi \alpha_{\max}}$, and by paying attention to formula 19, 20, approach velocity and elastic phase period obtains as follows.

$$\dot{\bar{\alpha}}_1 = \xi \sqrt{1 - \frac{4}{5\xi^2}}, \quad \tau_1 = \frac{1}{\xi} \int_0^1 \frac{d\bar{\alpha}}{\sqrt{1 - \frac{4}{5\xi^2} \bar{\alpha}^{5/2}}} \tag{28}$$

As it is known the approach velocity is zero at the end of elastic-plastic level. Thus, approach value and elastic-plastic level period will be as follows

$$\bar{\alpha}_2 = \sqrt{\frac{2}{3}\xi^2 - \frac{4}{45}} + \frac{1}{3}, \quad \tau_2 = \frac{2}{3}\left(\frac{\pi}{2} - Arc\tan\left(\frac{1}{\dot{\bar{\alpha}}_1}\right)\right) \tag{29}$$

And finally, impact period, velocity and approach velocity at the end of unloading phase would be as follows, respectively.

$$\tau_3 = \frac{1}{\sqrt{2}}\int_1^0 \frac{d\bar{v}}{\left[\bar{a}\left(1-\bar{v}^{3/2}\right)+\frac{2}{5}\left(1-\bar{v}^{5/2}\right)\right]^{1/2}}, \dot{\bar{v}}_f = \sqrt{2}\left[\bar{a}+\frac{2}{5}\right]^{1/2}, \dot{\bar{\alpha}}_3 = \sqrt{2}\left[\bar{\alpha}_2 - \frac{3}{5}\right]^{1/2} \tag{30}$$

In the above formula, $\bar{v} = (\bar{\alpha} - \bar{\alpha}_2 + 1), \bar{a} = (\bar{\alpha}_2 - 1)$. As it is evident from the above formula, force history value, impact period and non-dimensional approach velocity in every level can be predicted only by knowing $\xi$. For showing that how the above formula can be used to predict the mentioned cases in every level of elastic-plastic impact, it is necessary to obtain impact force history for some special cases. Now, in this part, the results of elastic-plastic impact analytical model, like non-dimensional force history and non-dimensional indentation history figures presented for 2 examples including the mentioned properties in Tables 3, 4 and maximum impact force and impact duration in every level will be compared with the results of the obtained formulation.

Table 3. material properties for impactor and plate for first example

| plate | impactor |
|---|---|
| Material: aluminum | Material: steel |
| $E_p = 70Gpa$ | $E_i = 206.85Mpa$ |
| $\upsilon_p = 0.34$ | $\upsilon_i = 0.29$ |
| $\rho_p = 2700kg/m^3$ | $\rho_p = 7800kg/m^3$ |
| $a,b,h: 1\times 0.667\times 0.15m$ | $R_i = 0.02m$ |
| $S_y = 324.028Mpa$ | $V_0 = 1m/s$ |

Table 4. Material properties for impactor an plate for the second example

| plate | impactor |
|---|---|
| Material: plump | Material: steel |
| $E_p = 17.9Gpa$ | $E_i = 206.85Mpa$ |
| $\upsilon_p = 0.45$ | $\upsilon_i = 0.29$ |
| $\rho_p = 11360kg/m^3$ | $\rho_p = 7800kg/m^3$ |
| $a,b,h: 1\times 2\times 0.2m$ | $R_i = 0.02m$ |
| $S_y = 82.74Mpa$ | $V_0 = 0.05m/s$ |

By paying attention to formulas 28 to 30, the following results are obtained in the shape of Tables 4 and 5. In every table the maximum impact force and also the maximum indentation for simulation and analysis models have been compared together.

Table 5.Comparing the results obtained from complete simulation and analytical model for the first example

| | $\overline{F}_1(\tau)$ | $\tau_1$ | $\overline{F}_2(\tau)$ | $\overline{\alpha}_2$ | $\tau_2$ | $\tau_3$ |
|---|---|---|---|---|---|---|
| By analytical model | 1 | 0.012 | 110.157 | 73.7711 | 1.288 | 1.428 |
| By Formulation | 1 | 0.011063 | 110.711 | 74.1404 | 1.05089 | 1.19295 |
| Error Percent | 0% | 7.808% | 0.5% | 0.5% | 18.4% | 15.69% |

Table 6.Comparing results obtained from complete simulation and analytical model for the second example

| | $\overline{F}_1(\tau)$ | $\tau_1$ | $\overline{F}_2(\tau)$ | $\overline{\alpha}_2$ | $\tau_2$ | $\tau_3$ |
|---|---|---|---|---|---|---|
| By analytical model | 1 | 0.06 | 20.5584 | 14.0389 | 1.304 | 1.63 |
| By Formulation | 1 | 0.05942 | 20.6162 | 14.0775 | 1.06701 | 1.39726 |
| Error Percent | 0% | 0.967% | 0.5% | 0.274% | 18.2% | 14.27% |

By the results obtained from comparing the complete impact simulation and the achieved analytical model, the following discussion can be ensued. (1) The plate under impact may be entered in the plastic phase even in very low energies. (2) From Tables 5 and 6 it can be concluded that an analytical model can present an exact prediction for maximum impact force and indentation. (3) Apparently, the analytical model in the prediction of impact period is not as successful as maximum impact force, which may be due to a lot of factors like plate boundary condition and dimensions. As it was implied that for times much longer than times needed by these waves to reach the boundaries and return to the point of impact, the lowest mode of projectile-target system predominates, for longer impact periods, the response is under the effect of shear and bending waves, and finally for the very longer periods of essential time for reaching waves to boundary condition and returning to impact point, the lower vibrational modes of projectile-target system are dominant. Thus, the effect of plate boundary condition and plate dimensions should be considered in conclusion. (4) If $\overline{\alpha}_2 > 1$, for achieving suitable responses, we should certainly use elastic-plastic impact law for the prediction of force history.

## 6. CONCLUSION

In this study, essential guidance was presented for the prediction of maximum impact force, approach velocity and impact duration in every 3 phases of elastic-plastic impact law for isotropic plates, beams, and rods. Dimensionless equations of motion of the plate were obtained by applying the Reissner-Mindlin plate theory which considers the first-order shear deformation and rotating inertia effects.

After obtaining the exact frequencies and related modeshapes, the forced vibration of the plate under applied load was presented by using Eigenvalue Expansion, and then Impact force history was obtained by using small time increment method. Then the effects of different dimensionless parameters on force-history have been examined and it is demonstrated that maximum impact force, approach velocity and impact period in every 3 levels of elastic-plastic impact can be predicted by just knowing a non-dimensional parameter. Finally, a criterion necessary for utilizing the effect of permanent indentation on impact response was developed. In this study both of elastic and elstoplastic laws are investigated and it is emphasized that in higher velocities, elastic-plastic law should be applied.

- ✓ We tried to present the equations in non-dimensional form. As a result, the effective parameters have decreased in impact response. The Figures are more general and include a broad scope of the impacts and also lead to a decrease in the number of the needed calculations.
- ✓ The most important effective parameter on the impact force history in elastic impact is $\lambda$ which affects both maximum force and the impact duration. By increasing the resulted force of the impact parameter, the impact also increases, but after enlarging this parameter, increase in the impact force, becomes slower.
- ✓ The aspect ratio, the thickness to length ratio and, also, the impactor's mass to the plate's mass ratio influence the impact force history and increase in these parameters leads to increase in the impact force and for the ratio of impactor mass to plate mass will decrease impact duration.
- ✓ In this paper, the effect of permanent indentation is, also, considered. In addition, a high limit is delivered for using Hertz theory, in a way that if $\overline{\alpha}_2 > 1$, for having proper responses, elastic-plastic impact must be used for predicting the impact history.
- ✓ Moreover, in this paper, the great effect of velocity on the impact history in elastic-plastic impact has been approved and it can be said that impact force increase is directly related to the velocity increase and also the impact response can be nearly predicted by having $\xi$. This has also been approved by the analysis method.

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