A Fourier Spectral Method To Solve High Order On-Surface Radiation Boundary Conditions in Electromagnetics

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***Abstract*—In this paper we develop a Fourier spectral method based on discrete Fourier transform to find the normal derivative of the electromagnetic field on the surface of the scatterer. The method involves choosing a high order local boundary conditions in frequency-domain and implementing it on the surface of the scatterer to find the normal derivative of the field numerically for applications such as the calculation of the radar cross-section and the surface current. We consider the two-dimensional problem for a perfectly conducting cylinder. Its numerical implementation and its performance in a wide range of frequency problems are demonstrated and compared to the frequency-domain integral equation for the scattered field. The advantage of the new method is that the On- Surface Radiation Boundary Conditions (OSRBCs) is applicable to a wide range of frequencies. A series of numerical tests demonstrate the accuracy and efficiency of these conditions to a wide range of frequencies. Both the exact solutions as well as the high order local boundary conditions solutions are compared.**

***Index Terms*—Radiation boundary conditions, scattering, frequency-domain analysis, numerical analysis, discrete Fourier transform**

I. Introduction

The development of efficient electromagnetics numerical techniques for the simulation of the electromagnetism scattering problems is a vast area of research and has gained a considerable amount of attention during the last four decades. This is due to new technological requirements and scientific applications such as electromagnetic waves scattered around an antenna, radar cross sections and surface current [1] and [2]. Many difficulties limit the application of classical numerical approaches. The first obstacle is related to the open domain where many wave propagation problems are described in the unbounded domain. This makes it difficult to extend the computational domain to the far field due to the dissipative and dispersive nature of the direct numerical method [3]. Moreover, relevant calculations depend on the accuracy of the radiation boundary conditions (RBCS) and are linked to the fact that the wavelength of the incident field is smaller than the characteristic size of the scatterer [4].

The past two decades had seen some accurate formulations to simulate scattering from two-dimensional obstacles. As an example, the Perfectly Matched Layer (PML) [5] and [6]. However, the computational effort needed for this method limit their application range even if it provides some valuable results. On the other side PML cannot be used as an on surface conditions because it has to be placed far away from the surface of the scatterer and we need to derive expression for normal derivative on the surface of the scatterer.

An alternative formulation is the On-Surface Radiation Boundary Condition (OSRBC) which is considered as the frontier between classical methods and asymptotics techniques [4]. It was introduced in the middle of the eighties by Kriegsmann and Tavlove [7]. It is an approximate technique that applies local artificial boundary conditions [8]–[10] directly on the surface of the scatterer to determine the normal derivative. In these papers, the authors also introduce the basic method to compute the electromagnetic scattered filed from a two-dimensional infinite cylinder. This method lead to numerical solution of a set of partial differential equations set over the surface of the scatterer Γ (or on contour for a 2-D object). Specifically, this principle involves the calculation of normal derivatives on Γ. Unfortunately, the closer one brings the radiation boundary to the scatterer, the more precise the radiation condition should be. In [7] the authors used either the first or second order radiation conditions. In addition, their work was limited to high-frequency problems ($k \geq 5$). In this work, we present an arbitrary order OSRBC in frequency domain in two dimensions to a wide range of frequencies.

In addition, Kriegsmann and Moore [11] have studied and applied OSRBC to a class of scattering problems in acoustics. Jones and Kriegsmann [12] has studied the effectiveness of OSRBC for a nonhomogeneous medium of an exterior problem and a boundary of varying curvature. This work was done in two dimensions and for high frequencies.

In a sequence of works developed by [13] and [14], the authors take into consideration arbitrary order boundary conditions for wave equations that use only a local operator on

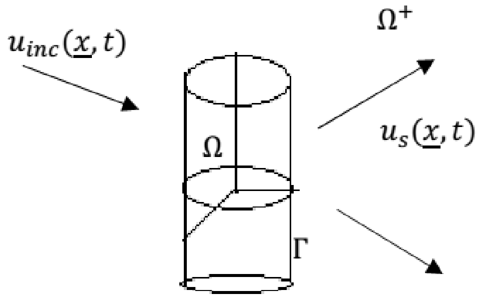

Fig. 1. Scattering Problem Configuration

the boundary. In these works, they removed the use of higher order radial derivatives found in the work of Bayliss and Turkel's boundary conditions [8]. In particular the work found in [13] replaces the difficulties using only higher order time derivatives which are easier to implement. The work presented here uses these results to formulate the on-surface radiation conditions. This forms the basis for the implementation of OSRBC in two dimensions proposed in this paper. In essence, we start with the introduction of two-dimensional Dirichlet problems governed by wave equations in the exterior domain. Then, we describe the high order local boundary conditions in frequency domain in two dimensions. Then, we apply the Fourier spectral method based on discrete Fourier transform in order to solve numerically for the normal derivative expressions ($\frac{\partial u}{\partial n}$) on the surface of the scatterer. This eliminates choosing different types of finite difference implementations. We show the accuracy and precision of this new high order local boundary conditions for calculation the surface current and the radar cross section for circular cross section compare to the exact calculation using integral equation methods. Finally, we verify our proposed results using numerical simulation.

II. The OSRBCs Formulation (TM Polarization)

To describe the wave propagation problem, we assume that there is a TM wave field (E) illuminating a perfect conducting cylinder with its cross section in the $x - y$ plane and its axis along z direction as shown in Fig. 1. This leads to a second order scalar wave equation.

Let $\underline{x} = (x, y)$ a point of R^2, $u_{inc}(\underline{x}, t)$ denotes the incident field and $u_s(\underline{x}, t)$ denotes the scattered field. The scattered field (u_s) to the cylinder which satisfies the so called wave equation.

$$\Delta u_s - \frac{1}{c^2}\frac{\partial^2 u_s}{\partial t^2} = 0 \qquad \text{in } \Omega^+ \tag{1}$$

For the well-posedness of the problem the field $u_s(\underline{x}, t)$ must also satisfy a boundary condition at the surface (Γ) of the cylinder as well as the sommerfeld radiation condition at infinity

$$\lim_{|r|\to\infty} |r|^{\frac{-1}{2}} \left(\frac{\partial}{\partial r} - ik\right) u_s = 0$$

Because equation (1) is of second order in space, we need two boundary conditions. The first boundary condition is the Dirichlet (perfect conductor) boundary condition. Here, the scatterer is assumed to be perfectly conducting on the Γ, so the field satisfies:

$$u_s(\underline{x}, t) = -u_{inc}(\underline{x}, t) \qquad \text{on } \Gamma$$

A second condition for two dimensions is a sequence of recursively defined boundary conditions. Typically it is applied to a circle exterior to the cylinder at a finite distance. In the OSRBC procedure we directly apply these boundary conditions on the surface of the scatterer itself. The authors of [13] derived higher order Radiation Boundary Conditions (RBCs) that are both asymptotically exact and easy to implement. The derivation of this boundary condition originates from the [8]. The higher order boundary conditions are expressed in the polar coordinate notation for two dimensions in frequency domain, as follows:

$$\frac{iw}{c}\hat{u} + \frac{\partial \hat{u}}{\partial r} + \frac{1}{2r}\hat{u} = \hat{\omega}_1 \tag{2}$$

$$\frac{iw}{c}\hat{\omega}_j + \frac{j}{r}\hat{\omega}_j = \frac{(j-\frac{1}{2})^2}{4r^2}\hat{\omega}_{j-1} + \frac{1}{4r^2}\frac{\partial^2 \hat{\omega}_{j-1}}{\partial \theta^2} + \hat{\omega}_{j+1} \tag{3}$$

where $\hat{u}(r \in \Gamma, \theta)$, $\frac{\partial \hat{u}}{\partial r}(r \in \Gamma, \theta)$ and $\hat{\omega}_j(r \in \Gamma, \theta)$ are auxiliary functions defined recursively and known as remainders, where we have set:

$$\hat{\omega}_0 = 2\hat{u}_s$$

When we set $\hat{\omega}_1 = 0$ the condition reduces to the well-known Bayliss and Turkel condition.
By direct computation, it has been proven in [13] for two-dimensional that:

$$\hat{\omega}_j = O(r^{-2j+\frac{1}{2}})$$

The equation (3) can be solved to find $\hat{\omega}_1$ in terms of $\hat{u}_s$. Using remainders of order (p) and set for order $(p+1)$ as higher to be as follows:

$$\forall j > p, \ \hat{\omega}_j = 0$$

Equations (2), (3) can be solved to find the radial derivative $\frac{\partial \hat{u}}{\partial r}$, which represents the normal derivative $\frac{\partial \hat{u}}{\partial n}$ at the surface of the scatterer. Using this result we can calculate the radar cross section (RCS) of a scatterer and the surface current (J) using the on service radiation boundary conditions. The scattered field $(\hat{u}_s(\underline{x}))$ at some distance from the scatterer boundary Γ is given by:

$$\hat{u}_s(\underline{x}) = \int_\Gamma \left[G(\underline{x}, \underline{y})\frac{\partial \hat{u}_s(\underline{y}))}{\partial n} - \hat{u}_s(\underline{y})\frac{\partial G(\underline{x}, \underline{y})}{\partial n}\right] ds \tag{4}$$

Where $G(\underline{x}, \underline{y})$ is the free space Green's function given by:

$$G(\underline{x}, \underline{y}) = -(i/4)H_0^1(kd)$$

and $d = |\underline{x} - \underline{y}|$, $\frac{\partial}{\partial n}$ is the outward normal derivative on Γ, $\underline{y}$ is on Γ, $\underline{x}$ is some distance from Γ, $\hat{u}_s$ is the Fourier transform of u_s, and k is the wave number. Thus, the scattered field is completely determined when the normal derivative $(\frac{\partial \hat{u}_s}{\partial n})$ is calculated. It is known that the far field expansion of (4) for two dimensional can be written as:

$$\hat{u}_s(\underline{x}) = A_0 \frac{e^{ikd}}{\sqrt{d}} \tag{5}$$

Where the term A_0 in (5) is given by:

$$A_0 = \frac{e^{j\pi 4}}{\sqrt{8k\pi}} \int_\Gamma \left[\frac{\partial \hat{u}_s}{\partial n} - jkcos\delta \hat{u}_{inc} \right] e^{-jk\psi} ds \tag{6}$$

Where $cos\delta = \underline{\hat{x}}.\hat{n}$ and $\psi = \underline{\hat{x}}.\underline{y}$. The RCS can be calculated using A_0:

$$RCS = 2\pi R_1 \left| A_0 \right|^2 \tag{7}$$

Another application is the calculation of the surface current of a scatterer. It is given by:

$$|J| = \left| \frac{i}{k} \frac{\partial}{\partial n} [\hat{u}_{inc} + \hat{u}] \right|_\Gamma \tag{8}$$

III. Numerical Implementation

The main computational challenge in this OSRBC is the implementation of equations (2) and (3) on the boundary. Specifically equation (3) has a second derivative term for every order of the boundary conditions. Further careful work at this reveals that one can exploit periodicity of the solution in θ. In this paper, we use the Fourier spectral method based on discrete Fourier transform (DFT) to solve the higher order local boundary conditions in two-dimensional to calculate the normal derivative [15]. The Fourier spectral method based on discrete Fourier transform (DFT) is a very powerful tool to solve partial differential equations with high precision using standard computers.

The $\hat{u}(r \in \Gamma, \theta)$, $\frac{\partial \hat{u}}{\partial r}(r \in \Gamma, \theta)$ and $\hat{\omega}_j(r \in \Gamma, \theta)$ in (2) and (3) can be written:

$$\begin{aligned} \hat{u}(r \in \Gamma, \theta) &= \sum_{m=-N/2}^{N/2-1} \alpha_m(r) e^{im\theta} \\ \frac{\partial \hat{u}}{\partial r}(r \in \Gamma, \theta) &= \sum_{m=-N/2}^{N/2-1} \gamma_m(r) e^{im\theta} \\ \hat{\omega}_j(r \in \Gamma, \theta) &= \sum_{m=-N/2}^{N/2-1} \alpha_m^{(j)}(r) e^{im\theta} \end{aligned}$$

Using the inverse discrete Fourier transform (IDFT) the $\alpha_m(r)$, $\gamma_m(r)$ and $\alpha_m^{(j)}(r)$ can be calculated as follows:

$$\begin{aligned} \alpha_m(r) &= \frac{1}{N} \sum_{l=-N/2}^{N/2-1} \hat{u}(r \in \Gamma, \theta_l) e^{-im\theta_l} \\ \gamma_m(r) &= \frac{1}{N} \sum_{l=-N/2}^{N/2-1} \frac{\partial \hat{u}}{\partial r}(r \in \Gamma, \theta_l) e^{-im\theta_l} \\ \alpha_m^{(j)}(r) &= \frac{1}{N} \sum_{l=-N/2}^{N/2-1} \hat{\omega}_j(r \in \Gamma, \theta_l) e^{-im\theta_l} \end{aligned}$$

From the boundary conditions, $\hat{\omega}_0(r \in \Gamma, \theta_l) = 2\hat{u}(r \in \Gamma, \theta_l)$ based on that :

$$\alpha_m(r) = \frac{1}{2} \alpha_m^{(0)}(r)$$

For simplicity, we will use the following notations:

$$\begin{cases} \alpha_m(r) &= \alpha_m &= \frac{1}{2}\alpha_m^{(0)} \\ \alpha_m^{(j)}(r) &= \alpha_m^{(j)} \\ \gamma_m(r) &= \gamma_m \end{cases}$$

Then, the equation (3) becomes:

$$\begin{aligned} &\frac{iw}{c} \sum_{m=-N/2}^{N/2-1} \alpha_m^{(j)} e^{im\theta} + \frac{j}{R} \sum_{m=-N/2}^{N/2-1} \alpha_m^{(j)} e^{im\theta} \\ &= \frac{(j-\frac{1}{2})^2}{4R^2} \sum_{m=-N/2}^{N/2-1} \alpha_m^{(j-1)} e^{im\theta} - \frac{m^2}{4R^2} \sum_{m=-N/2}^{N/2-1} \alpha_m^{(j-1)} e^{im\theta} \\ &\quad + \sum_{m=-N/2}^{N/2-1} \alpha_m^{(j+1)} e^{im\theta} \end{aligned} \tag{9}$$

The above equation can be simplified as:

$$\frac{iw}{c}\alpha_m^{(j)} + \frac{j}{R}\alpha_m^{(j)} = \frac{(j-\frac{1}{2})^2}{4R^2}\alpha_m^{(j-1)} - \frac{m^2}{4R^2}\alpha_m^{(j-1)} + \alpha_m^{(j+1)}$$

That is, $\forall m$:

$$\frac{(j-\frac{1}{2})^2 - m^2}{4R^2}\alpha_m^{(j-1)} = \left(\frac{iw}{c} + \frac{j}{R} \right) \alpha_m^{(j)} - \alpha_m^{(j+1)}$$

We assume that $X^{(j)} = [\alpha_{\frac{-N}{2}}^{(j)}, \cdots, \alpha_{\frac{N}{2}-1}^{(j)}]^T$ is defined:

$$H_j X^{(j-1)} = \left(\frac{iw}{c} + \frac{j}{R} \right) X^{(j)} - X^{(j+1)}$$

Where H_j is a $(N) \times (N)$ matrix given by:

$$H_j = \frac{1}{4R^2} \begin{pmatrix} a_{j\frac{-N}{2}} & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & a_{j0} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & a_{j\frac{N}{2}-1} \end{pmatrix}$$

With:

$$\begin{aligned} a_{j\frac{-N}{2}} &= (j-\tfrac{1}{2})^2 - (\tfrac{-N}{2})^2 \\ a_{j0} &= (j-\tfrac{1}{2})^2 \\ a_{j\frac{N-1}{2}} &= (j-\tfrac{1}{2})^2 - (\tfrac{N}{2}-1)^2 \end{aligned}$$

As a consequence,

$$X^{(j-1)} = \left(\frac{iw}{c} + \frac{j}{R} \right) H_j^{-1} X^{(j)} - H_j^{-1} X^{(j+1)}$$

Note that H_j is diagonal matrix which makes it easy to deal with numerically, and in particular to invert, and this feature will considerably improve the computational time.

Knowing that for each and every j: $\hat{\omega}^j = O(r^{\frac{1}{2}-2j})$. We can consider that for fixed arbitrarily index depending on the desired accuracy the remainder becomes so small that it can be considered to be 0. Let us call X^p the last non-zero remainder, that is all remainder X^j with $j > p$ is neglected and assumed to be zero. Next to solve (3) numerically and find the X^j's remainders [16], we denote:

$$X^{j-1} = H_j^{-1}\left[\left(\frac{iw}{c}+\frac{j}{R}\right)X^j - X^{j+1}\right], \quad \forall j < p$$

For the case when $j = p$:

$$X^{p-1} = \left(\frac{iw}{c}+\frac{p}{R}\right)H_p^{-1}X^p$$

To proceed further:

$$X^p = P_p X^p \quad and \quad X^{p-1} = P_{p-1}X^p$$

where:

$$P_p = I_{N+1} \quad and \quad P_{p-1} = \left(\tfrac{iw}{c}+\tfrac{p}{R}\right)H_p^{-1}$$

Below, we prove that, any remainder X^j can be expressed numerically with respect to the last one: X^p. Let's assume this is true for some $q < p$ and all order from q to p then:

$$\begin{array}{rcl} X^{q-1} & = & H_q^{-1}\left[\left(\frac{iw}{c}+\frac{j}{R}\right)X^q - X^{q+1}\right] \\ & = & H_q^{-1}\left[\left(\frac{iw}{c}+\frac{j}{R}\right)P_qX^p - P_{q+1}X^p\right] \\ & = & \underbrace{(H_q^{-1}(\dfrac{iw}{c}+\dfrac{j}{R})P_q - H_q^{-1}P_{q+1})}_{P_{q-1}}X^p \end{array}$$

Therefore,we proved by induction that for any integer $q \in \{1;2;\ldots;p\}$:

$$X^q = P_qX^p$$

And,

$$\left\{\begin{array}{rcl} X^1 & = & P_1X^p \\ X^0 & = & P_0X^p \end{array}\right.$$

From the above derivation, we find a recursive definition of the matrices P_q's:

$$\left\{\begin{array}{rcl} P_p & = & I_N \\ P_{p-1} & = & \left(\frac{iw}{c}+\frac{p}{R}\right)H_p^{-1}X^p \\ P_{q-1} & = & (H_q^{-1}(\frac{iw}{c}+\frac{j}{R})P_q - H_q^{-1}P_{q+1}) \end{array}\right.$$

Where $q = 2;\ldots;p-1$. Now, we can compute by backward iteration the matrices P_0 and P_1 such that:

$$X^0 = \frac{1}{2}\alpha_m^{(0)} = P_0X^p \quad \Rightarrow \quad X^p = P_0^{-1}\alpha_m^{(0)}$$

And:

$$X^1 = P_1X^p = P_1P_0^{-1}\alpha_m^{(0)}$$

Which, with the additional notation $Y = [\gamma_{\frac{-K}{2}}, \cdots, \gamma_{\frac{K}{2}}]^T$ and using (2) the solution is:

$$Y = X^{(1)} - \frac{1}{2}\left(\frac{iw}{c}+\frac{1}{2r}\right)X^{(0)}$$

That is:

$$Y = \left(P_1P_0^{-1} - \frac{1}{2}\left(\frac{iw}{c}+\frac{1}{2r}\right)\right)\alpha_m^{(0)}$$

Using the discrete Fourier transform (DFT) formula, we can find the normal derivative in frequency domain.

IV. RESULTS

This section demonstrate the effectiveness and usefulness of the application of the OSRBCs developed in this paper to the perfectly conducting circular cylinder of radius $R = 1$. The circular cylinder illuminated by a TM polarization plane wave.

$$\hat{u}_{inc} = e^{jkRcos\theta} \tag{10}$$

We developed a series of numerical tests to compute the radar cross sections (RCS) and the surface current (J) using the frequency domain OSRBCs and compared with the exact solution obtained using the integral equation method. Fig. 2 compares the computed results for the surface current (J) obtained using (8) and the radar cross section (RCS) using (7) for $k = 10$ and $P = 1$. Fig. 3 compares the computed results for the surface current (J) obtained using (8) and the radar cross section (RCS) using (7) for $k = 5$ and the order of the boundary condition $p = 2$. Fig. 4 compares the computed results for the surface current (J) obtained using (8) and the radar cross section (RCS) using (7) for $k = 1$ and the order of the boundary condition $p = 2$. Fig. 5 compares the computed results for the surface current (J) obtained using (8) and the radar cross section (RCS) using (7) for $k = 0.01$ and the order of the boundary condition $p = 2$. Fig. 6 compares the computed results for the surface current (J) obtained using (8) and the radar cross section (RCS) using (7) for $k = 0.01$ and the order of the boundary condition $p = 20$. Fig. 7 compares the computed results for the surface current (J) obtained using (8) and the radar cross section (RCS) using (7) for $k = 0.01$ and the order of the boundary condition $p = 81$.The above figures show that the OSRBCs is close to the exact solution and leads to high accuracy for high frequency $k \geqslant 1$ for the order of the boundary condition ($p = 2$). For the case of low frequency e.g $k = 0.01$, we should increase the order of the boundary conditions ($p = 81$) until the results become order independent. It is interesting to note that the error are decreasing by increasing the order of the boundary condition at low frequency as seen in the Fig. 7 compare to the results in Fig. 5 and Fig. 6.

V. CONCLUSION

This paper presented a new procedure for implementing the OSRBC in frequency domain using the high order local boundary conditions introduce in [13] for two dimensions. As shown the analysis and the numerical implementations calculate accurately and efficiently the normal derivatives ($\frac{\partial \hat{u}}{\partial n}$) on the boundary of the scatterer ($R = 1$). Numerical examples are provided for the simulations of the surface current and the radar cross section for wide range of frequencies. Fourier spectral method based on discrete Fourier transform techniques used to implement the numerical analysis. For high frequencies, as expected the order of the OSRBC (p) can be as low as 2. For lower frequencies has to be substantially higher > 80 which is an order never investigated in the literature.

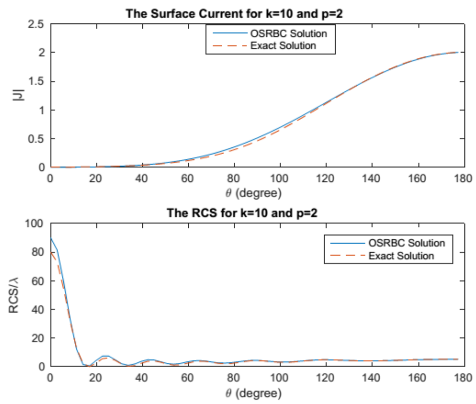

Fig. 2. Comparison of surface current (J) and radar cross section (RCS) for circular cylinder, TM case, calculated using the OSRBCs method, showing convergence to exact solution for $k = 10$ and $p = 2$

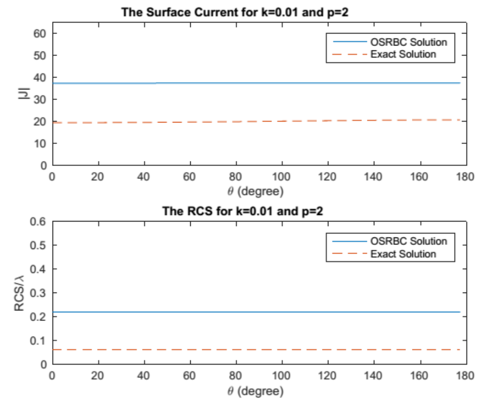

Fig. 5. Comparison of surface current (J) and radar cross section (RCS) for circular cylinder, TM case, calculated using the OSRBCs method, showing convergence to exact solution for $k = 0.01$ and $p = 2$

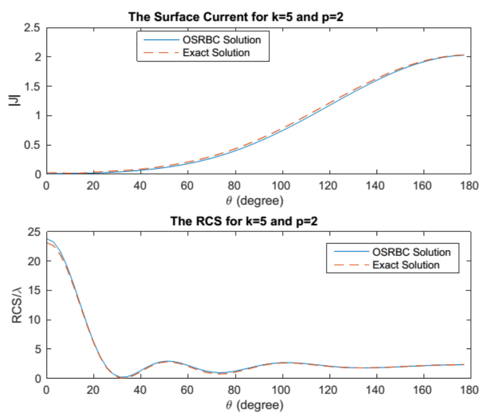

Fig. 3. Comparison of surface current (J) and radar cross section (RCS) for circular cylinder, TM case, calculated using the OSRBCs method, showing convergence to exact solution for $k = 5$ and $p = 2$

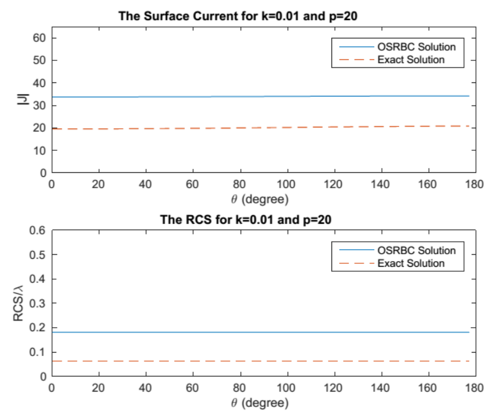

Fig. 6. Comparison of surface current (J) and radar cross section (RCS) for circular cylinder, TM case, calculated using the OSRBCs method, showing convergence to exact solution for $k = 0.01$ and $p = 20$

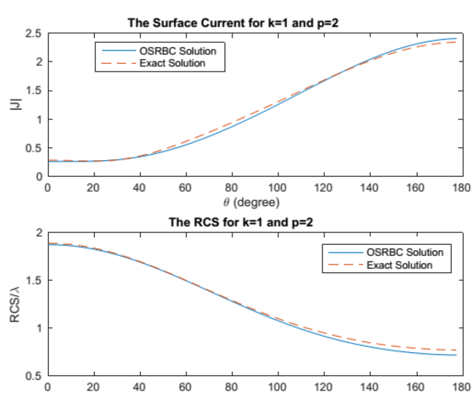

Fig. 4. Comparison of surface current (J) and radar cross section (RCS) for circular cylinder, TM case, calculated using the OSRBCs method, showing convergence to exact solution for $k = 1$ and $p = 2$

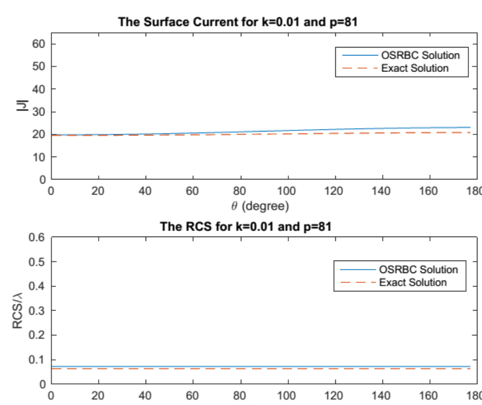

Fig. 7. Comparison of surface current (J) and radar cross section (RCS) for circular cylinder, TM case, calculated using the OSRBCs method, showing convergence to exact solution for $k = 0.01$ and $p = 81$

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